The function was

$$f(x) = 12\cos(3x) + 500x + 17x^{13}.$$

Let's try to find

$$(f^{-1})'(12).$$

Note: $f(0) = 12\cos(0) + 0 + 0$

$= 12,$

so $f^{-1}(12) = 0.$

Amazing fact: We can find a formula for $(f^{-1})'$ at a point without actually computing f^{-1}!

Recall: $f^{-1}(f(x)) = x$.

Assume both f and f^{-1} are differentiable.

Then since $f^{-1}(f(x)) = x$, differentiating both sides and using the chain rule, we get

$$\frac{d}{dx}\left(f^{-1}(f(x))\right) = \frac{d}{dx}(x)$$

$$= (f^{-1})'(f(x)) \cdot f'(x) = 1.$$

Dividing by $f'(x)$, we get

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

With $y = f(x)$, $x = f^{-1}(y)$,

so we can rewrite the formula as

$$\boxed{(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}}$$

* you can only use this formula when $f'(f^{-1}(y)) \neq 0$

Example 1: Apply the formula to find $(f^{-1})'(12)$ when $f(x) = 12\cos(3x) + 500x + 17x^{13}$.

Formula:

$$(f^{-1})'(12) = \frac{1}{f'(f^{-1}(12))}.$$

Since $f(0) = 12$, $f^{-1}(12) = 0$.

So $(f^{-1})'(12) = \dfrac{1}{f'(0)}$.

We've already calculated

$$f'(x) = -36\sin(3x) + 500 + 221x^{12},$$

which gives us

$$f'(0) = 0 + 500 + 0 = 500,$$

and

$$\boxed{(f^{-1})'(12) = \frac{1}{500}}$$

Example 2: Show that

$$f(x) = -21x - 3x^7 + \cos(\pi x)$$

is invertible, and find

$$(f^{-1})'(-25).$$

Observe

$$f'(x) = -21 - 21x^6 - \pi\sin(\pi x)$$

$$\leq -21 + 0 - \pi\sin(\pi x)$$

since $-21x^6 \leq 0$.

Now $-1 \leq \sin(\pi x) \leq 1$ and multiplying through by $-\pi$ gives $\pi \geq -\pi \sin(\pi x) \geq -\pi$.

Use this to get

$$f'(x) \leq -21 - \pi \sin(\pi x)$$
$$\leq -21 + \pi$$
$$\leq -17 < 0$$

since $\pi < 4$.

We've shown that f is decreasing and so is invertible. Then

$$(f^{-1})'(-25) = \frac{1}{f'(f^{-1}(-25))}$$

according to the formula.

Since $f(1) = -25$, $f^{-1}(-25) = 1$,

so

$$(f^{-1})'(-25) = \frac{1}{f'(1)}$$

$$f'(x) = -21 - 21x^6 - \pi \sin(\pi x)$$

$$f'(1) = -21 - 21 - \pi \sin(\pi)$$

$$= -42.$$

Then

$$\boxed{(f^{-1})'(-25) = \frac{1}{-42}}$$

Section 6.2*

Logarithms (blue-ish pages after 6.4).

First, $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$

except for $n = -1$!

What do we do then?

Fundamental Theorem of Calculus
(2 parts)

Let f be continuous on an open interval containing $[a,b]$.

Then

1) $$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

for all $a \leq x \leq b$. In particular, $g(x) = \int_a^x f(t)\,dt$ has a derivative.

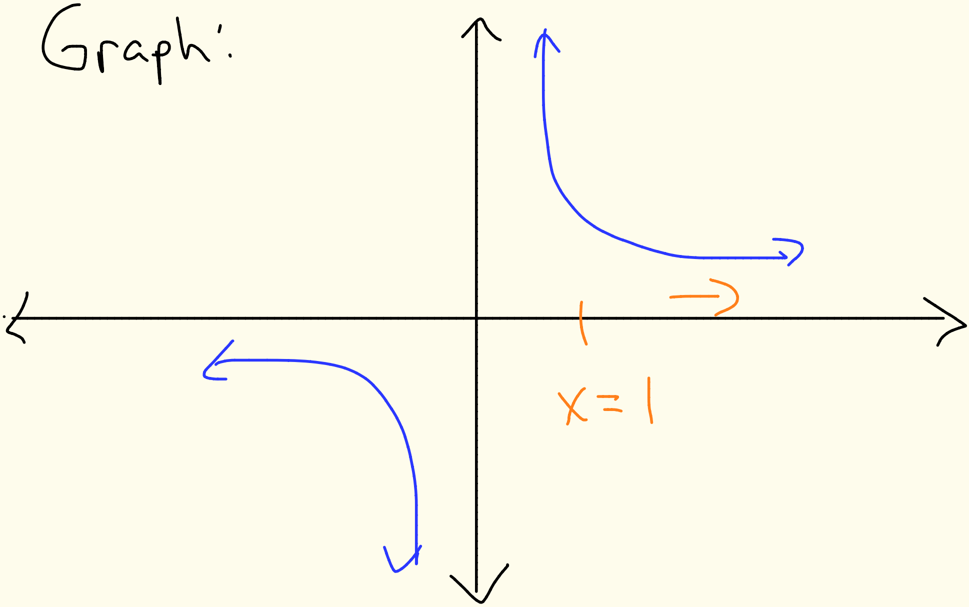
2) If h is any antideriviative of f that is continuous at $x=a$ and $x=b$, then

$$\int_a^b f(x)\,dx = h(b) - h(a)$$

Set $f(x) = \frac{1}{x} = x^{-1}$.

Then f is continuous on an open interval containing $[1, x]$ where $x > 1$.

Graph:



Define $\boxed{g(x) = \int_1^x \frac{1}{t}\,dt.}$

By the Fundamental Theorem of Calculus (part 1),

$$g'(x) = \frac{1}{x}.$$

Recall $\ln(1) = 0$. Observe

$$g(1) = \int_1^1 \frac{1}{t}\,dt = 0.$$ ✓

Also, $\ln(xy) = \ln(x) + \ln(y)$ $(x, y > 0)$. If $x, y \geq 1$,

$$g(xy) = \int_1^{xy} \frac{1}{t}\,dt$$

and

$$g(x) + g(y) = \int_1^{x} \frac{1}{t}\,dt + \int_1^{y} \frac{1}{t}\,dt$$

Suppose y is fixed and x is variable. Then by the chain rule,

$$\frac{d}{dx}\left(g(xy)\right) = g'(xy)\cdot \underbrace{y}_{=\frac{d}{dx}(xy)}$$

Since $g'(x) = \dfrac{1}{x}$,

$g'(xy) = \dfrac{1}{xy}$. So

$$g'(xy)\cdot y = \frac{1}{x\cancel{y}}\cdot \cancel{y} = \frac{1}{x}$$

$$\frac{d}{dx}\left(g(x)+g(y)\right)$$

$$= g'(x)+0 = g'(x) = \frac{1}{x}$$

Since y is fixed.

So we know

$$\boxed{\frac{d}{dx}\left(g(xy)\right) = \frac{d}{dx}\left(g(x)+g(y)\right)}$$

This shows

$$\boxed{g(xy) = g(x) + g(y) + C}$$

for some constant C that is independent of y.

If $x = 1$,

$$g(y) = g(1) + g(y) + C.$$

But $g(1) = \int_1^1 \frac{1}{t}\,dt = 0$,

so $g(y) = g(y) + C$ and $C = 0$.

We've shown

$$\boxed{g(xy) = g(x) + g(y)}$$

for $x, y \geq 1$, and you can check, using the same methods, that for all $x, r \geq 1$,

$$\boxed{g(x^r) = r\,g(x)}.$$

(Recall $\ln(x^r) = r\ln(x)$).

Extend g to x in $(0,1)$ by defining

$$g(x) = -\int_x^1 \frac{1}{t}\,dt$$

$$= -\left(-\int_1^x \frac{1}{t}\,dt\right)$$

$$= \int_1^x \frac{1}{t}\,dt$$

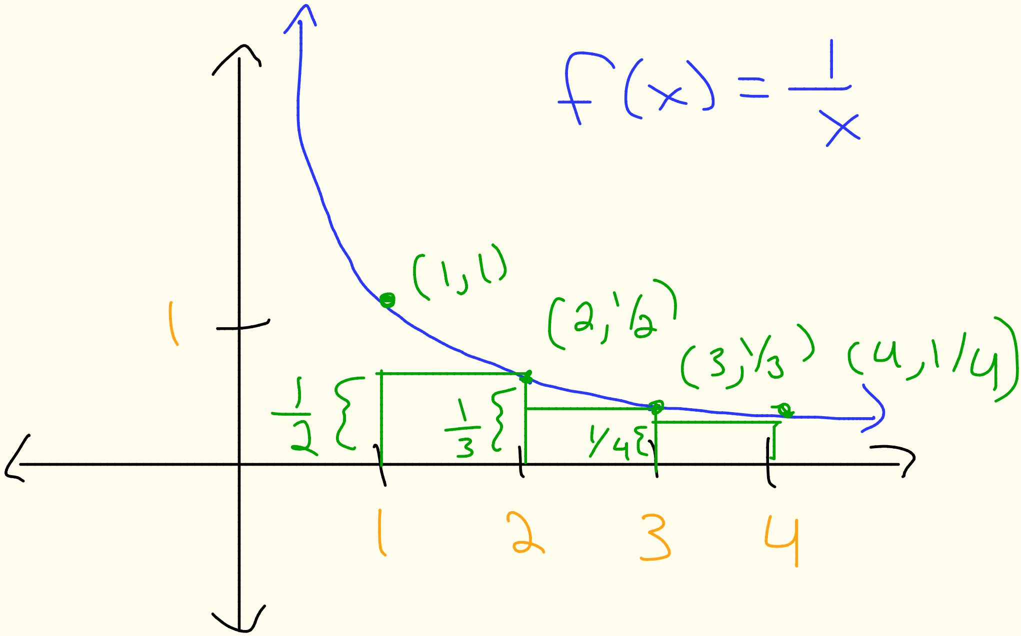
Define

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

for $x > 0$.

We want to define e as the number where $\ln(e) = 1$, but how do we know there is such a thing?

Proof by picture



Area of all the boxes drawn is $= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$.

$$\int_1^4 \frac{1}{x}\,dx > (\text{Area of boxes}) > 1$$

Then $\ln(4) > 1$, $\ln(1) = 0$,

so by the Intermediate Value Theorem, there is a number between 1 and 4 such that ln evaluates to one at that number. We'll call this number e.

Observe that

$$\ln(e^x) = x\ln(e) \text{ (powers)}$$

$$= x \cdot 1 = x.$$

So e^x is the inverse

function of $\ln(x)$.

Let's let $f(x) = \ln(x)$.

Then $f^{-1}(x) = e^x$ and

by our inverse formula,

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

But $f'(x) = \frac{1}{x}$, so

$$(f^{-1})'(y) = \frac{1}{\left(\frac{1}{f^{-1}(y)}\right)} = f^{-1}(y) = e^y.$$

What you should know

1) $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

2) $\frac{d}{dx}(e^x) = e^x$

3) log properties: $\ln(x^r) = r\ln(x)$,

$\ln(xy) = \ln(x) + \ln(y)$

4) exponential properties:

$(e^{xy}) = (e^x)^y$, $e^{x+y} = e^x e^y$

5) $\ln(e^x) = x$, $e^{\ln(x)} = x$.

Example 3: Find $f'(x)$ if $f(x) = \ln(\tan(x))$.

$$\frac{d}{dx}(f(x)) = \left(\frac{d}{dx}\ln\right)\underbrace{(\tan(x))\sec^2(x)}_{\text{chain rule}}$$

$$= \frac{1}{\tan(x)} \cdot \sec^2(x)$$

since $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

$$= \boxed{\cot(x)\sec^2(x)}$$

Example 4: Find the derivative of $f(x) = e^{(x^3+1)}$.

$$\frac{d}{dx}(f(x)) = \underbrace{\left(\frac{d}{dx}e\right)(x^3+1) \cdot 3x^2}_{\text{chain rule}}$$

$$= \boxed{e^{(x^3+1)} \cdot 3x^2}$$

since $\frac{d}{dx}(e^x) = e^x$.

Example 5: Find

$$\int e^{\sin(x)} \cos(x)\, dx.$$

This means "find the most general antiderivative of $e^{\sin(x)} \cos(x)$.

The answer is $e^{\sin(x)} + C$, but why?

To see this, set

$u = \sin(x)$, (substitution)

$du = \cos(x)\,dx$. Then

$$\int \underbrace{e^{\sin(x)}}_{u} \underbrace{\cos(x)\,dx}_{du}$$

$$= \int e^u \, du = e^u + C$$

Example 6: $f(x) = x^x$, $x > 0$.

Find $f'(x)$.

$$f'(x) \neq x \cdot \left(x^{x-1}\right)$$

Since x is nonconstant – the power rule doesn't apply.

Write
$$\boxed{\begin{aligned} f(x) &= e^{\ln(x^x)} \\ &= e^{x\ln(x)} \end{aligned}}$$

by log properties.

Using the chain rule,

$$\frac{d}{dx}\left(e^{x\ln(x)}\right)$$

$$= e^{x\ln(x)} \cdot \frac{d}{dx}\left(x\ln(x)\right)$$

$$= e^{x\ln(x)}\underbrace{\left(x\cdot\frac{1}{x} + \ln(x)\right)}_{\text{product rule}}$$

$$= e^{x\ln(x)}\left(1+\ln(x)\right)$$

$$= \boxed{x^{x}\left(1+\ln(x)\right)}$$